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The most Probable Position of a Point determined from the Intersections of Three Straight Lines. By S. A. Saunder, M.A.

In the course of my work on the Moon I have frequently fixed the position of what may be termed a point of the second order by measuring its position angles from three points whose coordinates had been well determined. I have shown (*B.A.A. Memoirs*, vol. vii. pp. 61–65) that when the points are near the centre of the Moon's disc, and the distances between them are small, the errors involved by neglecting the effects of libration are also small, whilst the reduction of the measures is thereby much simplified.

In the course of these reductions I have had to consider what was the most probable position of a point so determined. I am not aware what is the practice of those who compute meteoric radiants, but a recent writer (*Monthly Notices*, vol. lxv. p. 238) has assumed that the radiant should be placed at the incentre of the triangle formed by these lines. As this can be correct only under very special circumstances, I have thought it might be worth while to call attention to the point.

Supposing the determinations of the lines to be of equal weight, the conditional equations should be so stated that equal residuals are equally probable, and this would seem to be effected, at all events in my work, by writing the equations in the form $x\cos\alpha + y\sin\alpha - p = 0$, so that the residuals represent the perpendiculars from the point finally determined upon the three straight lines. If a distant point were observed with a theodolite the equations would require to be differently weighted.

If we use trilinear coordinates, taking these straight lines as sides of the triangle of reference, these three residuals will be the coordinates α, β, γ of the point, and we have to determine their values so that $\alpha^2+\beta^2+\gamma^2$ may be a minimum.

A necessary condition is that

$$\alpha\delta\alpha + \beta\delta\beta + \gamma\delta\gamma = 0 \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

and

$$a\alpha + b\beta + c\gamma = 2\Delta \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

$\therefore$

$$a\delta\alpha + b\delta\beta + c\delta\gamma = 0 \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

From (1) and (3)

$$(c\alpha - a\gamma)\delta\alpha + (c\beta - b\gamma)\delta\beta = 0$$

and as α, β are now independent this gives

$$\frac{\alpha}{a} = \frac{\beta}{b} = \frac{\gamma}{c} = \frac{2\Delta}{a^2+b^2+c^2} \quad \text{by (2)}$$

If $u \equiv \alpha^2+\beta^2+\gamma^2$ it is easily shown by actual differentiation that

$$r = \frac{\partial^2 u}{\partial\alpha^2} = 2 + 2\frac{a^2}{c^2}, \quad s = \frac{\partial^2 u}{\partial\alpha\partial\beta} = 2\frac{ab}{c^2}, \quad t = \frac{\partial^2 u}{\partial\beta^2} = 2 + 2\frac{b^2}{c^2}$$

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and hence that r, t and $rt-s^2$ are necessarily positive, which complete the conditions for a minimum.

The point thus found is the symmedian point which coincides with the incentre only when the triangle is equilateral. It may be constructed graphically from any one of the following properties :

If A B C is the triangle, I the incentre, G the centroid, K the symmedian point, AK, BK, CK, cut the sides in D, E, F respectively, and bc is an antiparallel to BC, so that $A\hat{b}c = A\hat{B}C$.

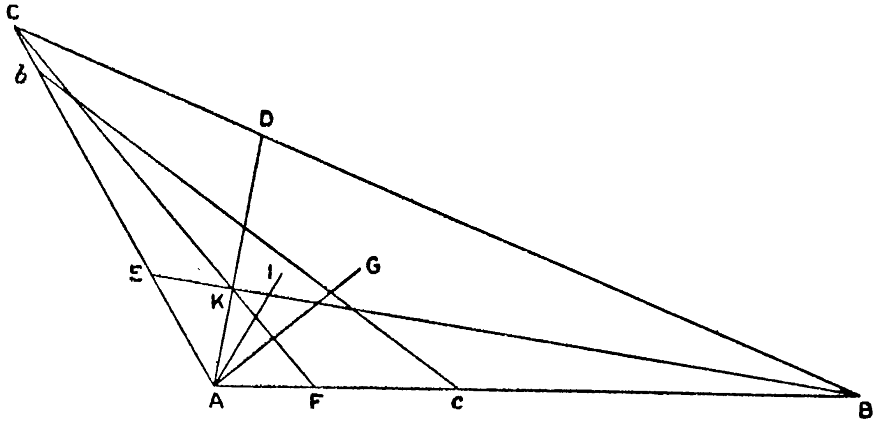
Then $\quad \text{BD} : \text{DC} :: c^2 : b^2$

$\text{AK bisects } bc$

and $\quad \hat{\text{I}}\text{AK} = \text{I}\hat{\text{A}}\text{G}$

with symmetrical relations for the other sides and angles.

This property of the symmedian point is not new, and the only object of this note is to call attention to its bearing upon the problem proposed.



If the determinations of the three lines are not of equal weights, let us suppose their weights to be l, m, n respectively. Then the condition required is that $l\alpha^2+m\beta^2+n\gamma^2$ should be a minimum ; whence, as before :

$$\frac{l\alpha}{a}=\frac{m\beta}{b}=\frac{n\gamma}{c}=\frac{2lmn\Delta}{mna^2+nlb^2+mlc^2}$$

And in the figure

$$\frac{\text{BD}}{\text{DC}}=\frac{mc^2}{nb^2},\quad \frac{\text{CE}}{\text{EA}}=\frac{na^2}{lc^2},\quad \frac{\text{AF}}{\text{FB}}=\frac{lb^2}{ma^2}$$

If the points c, b are so taken that $\dfrac{\text{A}c}{\text{A}b}=\dfrac{n\,.\,b}{m\,.\,c}$, bc is still bisected by AK.

Mr. H. C. Plummer has also pointed out to me that if G be

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the centre of gravity of weights l, m, n at A, B, C the angles IAK, IAG are still equal, thus completing the generalisation of the properties of the symmedian point.

The weights to be given depend, in my work, only on the numbers of times the directions have been measured. In the case of meteoric paths they would seem to depend upon the length of the observed path, its nearness to recognised stars, its distance from the radiant, the persistence of a trail, and possibly upon other causes. K will coincide with I when the weights l, m, n are proportional to a, b, c.

The Cartesian coordinates of K are easily written down if we remember that a, b, c are proportional to the sines of the angles between the lines; but the expressions are so long that in practice I prefer to effect each solution by least squares.